On the relations between fracture energy and physical observables in dynamic earthquake models

Andrea Bizzarri

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We explore the relationships between the fracture energy density ($E_G$) and the key parameters characterizing earthquake sources, such as the rupture velocity ($v_r$), the total fault slip ($u_{tot}$), and the dynamic stress drop ($\Delta \tau_d$). We perform several numerical experiments of three-dimensional, spontaneous, fully dynamic ruptures developing on planar faults of finite width, obeying different governing laws and accounting for both homogeneous and heterogeneous friction. Our results indicate that $E_G$ behaves differently, depending on the adopted governing law and mainly on the rupture mode (pulselike or cracklike, sub- or supershear regime). Subshear, homogeneous ruptures show a general agreement with the theoretical prediction of $E_G \propto \sqrt{1 - \left(\frac{v_r^2}{v^2}\right)}$, but for ruptures that accelerate up to supershear speeds it is difficult to infer a clear dependence of fracture energy density on rupture speed, especially in heterogeneous configurations. We see that slip pulses noticeably agree with the theoretical prediction of $E_G \propto u_{tot}^2$, contrarily to cracklike solutions, both sub- and supershear and both homogeneous and heterogeneous, which is in agreement with seismological inferences, showing a scaling exponent roughly equal to 1. We also found that the proportionality between $E_G$ and $\Delta \tau_d$, expected from theoretical predictions, is somehow verified only in the case of subshear, homogeneous ruptures with RD law. Our spontaneous rupture models confirm that the total fracture energy (the integral of $E_G$ over the whole fault surface) has a power law dependence on the seismic moment, with an exponent nearly equal to 1.13, in general agreement with kinematic inferences of previous studies. Overall, our results support the idea that $E_G$ should not be regarded as an intrinsic material property.


1. Introduction

[2] The so-called “fracture” energy density, $E_G$ (where $[E_G] = J/m^2$), is recognized to be one of the most important parameters in the context of the physics of the earthquake source and directly influences earthquake dynamics, since its value controls the rupture propagation and its arrest [Husseini et al., 1975; Schmeedes et al., 2010]. In addition, it affects radiation efficiency [e.g., Husseini and Randall, 1976; Venkataraman and Kanamori, 2004]. In recent years, many efforts have been made in order to retrieve its value from laboratory experiments [e.g., Wong, 1982; Lockner and Okubo, 1983] as well as from seismological inferences [McGarr et al., 2004; Tinti et al., 2005, and references therein] and to try to establish some analytical or empirical relations between $E_G$ and macroscopic physical observables, such as the scalar seismic moment ($M_0$), the rupture velocity of the propagating crack front ($v_r$), and stress drop.

[3] $E_G$ can be physically defined as the amount of energy (for unit fault surface) necessary to maintain an ongoing rupture which propagates on a fault (or alternatively, as the work done against the resistance to fault extension at the rupture tip). It is often called seismological fracture energy density and has been denoted with symbol $G$ (or $G_s$) in a large number of previous papers. In the framework of linear elastic fracture mechanics (LEFM) it has been associated with critical stress intensity factors [Irwin, 1957; Broberg, 1999; Tada et al., 2000] for different modes of crack propagation, which are material parameters and depend on the temperature and pressure conditions, grain size, etc. [e.g., Paterson and Wong, 2005]. The understanding of the earthquake energy budget (i.e., the quantification of the amounts of energy dissipated during coseismic ruptures by the production of new fracture surfaces, by seismic wave emission, thermal processes, etc.) is one of the fundamental open issues in earthquake source physics [e.g., Brown, 1998].

[4] From a mathematical point of view, on a specific point on the fault, we can define $E_G$ as the difference between the energy absorbed per unit area on the fault plane and the
work done against the frictional stress [e.g., Bizzarri and Cocco, 2006b] as

\[ E_G = \frac{d}{d_0} \int_0^d (\tau - \tau_{res})du, \]  

where \( d \) is the amount of cumulative slip (\( u \)) at which the value of the magnitude \( \tau \) (i.e., \( l^2 \)-norm) of the fault shear traction \( T \) reaches the residual level of friction, \( \tau_{res} \), attained after the completion of the stress release. Figure 1 shows a typical traction behavior for increasing cumulative fault slip and the geometrical interpretation of the previous definition of \( E_G \). Equation (1) implicitly assumes that \( \tau \) has a dependence on \( u \), which can be explicit, as for the slip-dependent friction laws [e.g., Ida, 1972] or implicit, as for other governing models. The quantity \( d \) in (1) can be associated with the characteristic distance \( d_0 \) in the context of the linear slip-weakening (SW) friction, or to its equivalent \( d_0^{eq} \) (see Cocco and Bizzarri [2002] for a detailed discussion) in the framework of the laboratory-derived rate- and state-dependent (RS) governing laws [e.g., Ruina, 1983] or, more generally, in the case of other nonlinear constitutive equations. Analogously, the quantity \( \tau_{res} \) in (1) can be associated with the kinetic level of friction, \( \tau_{f} \), in the case of the SW law, or to its equivalent \( \tau_{f}^{eq} \) [see Bizzarri and Cocco, 2003], in the case of RS laws. In general, we can regard \( \tau_{res} \) as the value that the fault friction attains when all the dissipative, chemophysical processes occurring during the coseismic breakdown phase are completed [Bizzarri and Cocco, 2006a, 2006b] (see Bizzarri [2009b] for a comprehensive review). As defined in (1), in general \( E_G \) is not a prior-imposed constitutive property but is determined by the dynamic time evolution of the total traction.

If the considered fault is governed by the linear SW law (see the dashed curve in Figure 1), then equation (1) is simply reduced to [see Palmer and Rice, 1973]

\[ E_G = \Delta \tau_b \frac{d_0}{2}, \]  

which depends only on constitutive parameters (\( \Delta \tau_b = \tau_u - \tau_f \) is the breakdown stress drop, with \( \tau_u \) being the upper yield stress); in this case \( E_G \) is a prior-imposed property. If \( \Delta \tau_b \) and \( d_0 \) are homogeneous over the whole fault surface, then also \( E_G \) is spatially constant, even if there are no theoretical
requirements for that. Moreover, when \( \tau_f \) is a well-defined level, it is apparent from (2) that \( E_G \) can be interpreted as the energy in excess of the energy required to sustain frictional sliding at shear level \( \tau_f \).

[6] In the framework of the RS laws, equation (2) can be rewritten as

\[
E_G \approx b \sigma_{\text{eff}}^n \left[ \ln \left( \frac{v_f}{v_0} \right) \right]^2 \frac{L}{2},
\]

where \( b \) is a constitutive parameter, \( \sigma_{\text{eff}}^n \) is the effective normal stress, \( L \) is the scale length for the evolution of the state variable, \( v_0 \) is the initial fault slip velocity, and \( v_f \) is its value (a priori unknown) after the breakdown process. For analytical details, see Bizzarri and Cocco [2003].

[7] When isotropic friction is not assumed, that is, when the fault shear traction vector \( T \) is not collinear to fault slip velocity vector \( \mathbf{v} \) (namely, when \( T \neq \left| |T||v||v| \right| \)), as is usually assumed in spontaneous dynamic earthquake models (Bizzarri and Cocco [2005], among others), equation (1) has been generalized as [Tinti et al., 2005]

\[
E_G = \int_0^{T_b} (T - T_{\text{res}}) \cdot \mathbf{v} \, dt,
\]

where \( T_b \) is breakdown duration (i.e., the time interval over which the stress release is realized), \( T_{\text{res}} \) is the residual shear traction vector (its Euclidean norm is the quantity \( \tau_{\text{res}} \) in (1)), \( t \) is the time and the bullet symbol indicates the scalar product. Tinti et al. [2005] term the quantity expressed by equation (4) breakdown work, even if, strictly speaking, it accounts also for the energy spent before the beginning the breakdown phase (i.e., during the possible early strengthening stage of the rupture, where the fault traction increases for increasing slip or slip velocity).

[8] The total fracture energy, \( U_G \) (where \( [U_G] = J \)) is the integral of \( E_G \) defined by equation (1) or (4) over the whole fault surface as

\[
U_G = \int_S E_G(\xi) \, d\xi,
\]

where \( \xi \) maps the fault surface \( \Sigma \). It is apparent that while the quantity \( E_G \) is a local estimate, which can be spatially variable as a consequence of the heterogeneous distribution of shear traction and slip velocity, \( U_G \) is a global estimate, which characterizes the whole seismic rupture event.

[9] Although some authors have considered the surface energy (i.e., the amount of energy spent in creating new sliding surfaces; in other words, the energy needed to break bonds) as the same physical quantity as the fracture energy [Wilson et al., 2005; Yoshioka, 1996], after the studies of Chester et al. [2005], Tinti et al. [2005], and Pittarello et al. [2008] it is now clear that surface energy is only a small fraction of the mechanical work absorbed on the fault [see also Lockner and Okubo, 1983]. Pittarello et al. [2008] also show that the most prominent fraction of \( E_G \), which in turn is not a negligible contribution to the earthquake energy budget [Venkataraman and Kanamori, 2004; Tinti et al., 2005; Cocco et al., 2006], is represented by heat produced by frictional sliding. Moreover, we emphasize that \( E_G \) is not only due to interfacial friction, but it is the sum of all energies associated with breakdown mechanisms.

2. Existing Scaling Relations for Fracture Energy

[10] From laboratory experiments of initially intact rock fracture and mode II shear failure on preexisting faults (precut samples) loaded by a two-axial apparatus, Ohnaka [2003] through his equation (22) proposes a linear dependence of \( E_G \) on the characteristic wave length \( \lambda_c \) of the topography of the sliding surface at which its self-similarity breaks down as

\[
E_G = 0.281 \tau_u \left( \frac{\Delta \tau}{\tau_0} \right)^{1.83} \lambda_c.
\]

In the context of a constitutive model different from the SW law, the quantities \( \tau_u \) and \( \tau_f \) have to be regarded as their equivalents, \( \tau_u^q \) and \( \tau_f^q \), respectively [see Bizzarri and Cocco, 2003].

[11] The obvious scale dependence of the parameter \( \lambda_c \) makes \( E_G \) also scale dependent. This scale dependence [see Ionescu and Campillo, 1999] is furthermore apparent from equation (1), where it is stated as an explicit dependence of the fracture energy density on the length scale \( d \). On the other hand, Otsuki [2007] found that the average fracture energy density is proportional to the length of a seismic rupture zone to the power of 0.56.

[12] By considering a slip pulse [Freund, 1979] obeying the position-weakening friction law introduced by Palmer and Rice [1973] (in this simplified form of the SW law, the traction linearly degrades with increasing spatial position of the rupture tip), Rice et al. [2005] through their equation (17), found, in two-dimensions (2D), a relationship between \( E_G \), the total cumulative slip \( u_{\text{tot}} \) developed during the time duration of the pulse \( (t_{\text{pulse}}) \), and the rupture velocity as

\[
E_G = \frac{G^2 u_{\text{tot}}}{\pi L_{\text{pulse}}} F(v_r),
\]

where \( L_{\text{pulse}} \) is the spatial length of the pulse, which can be approximated as \( L_{\text{pulse}} \approx \langle v_r \rangle t_{\text{pulse}} \) (where \( \langle v_r \rangle \) is the average rupture velocity), \( G \) is the rigidity of the medium, and the dimensionless function \( F(v_r) \) depends on the rupture modes and is defined as

\[
F(v_r) = \begin{cases} \frac{R}{\alpha_S(1 - \alpha_S)} & , \text{for mode II} \\ \frac{\alpha_P}{\alpha_S} & , \text{for mode III} \end{cases}
\]

[see also Rice et al., 2005, equation (11)]. In equation (8) \( R = 4\alpha_S \alpha_P - (1 + \alpha_S)^2 \) is 4 times the Rayleigh function, \( \alpha_S = \sqrt{1 - \left( \frac{\nu_P^2}{\nu_S^2} \right)^2} \) and \( \alpha_P = \sqrt{1 - \left( \frac{\nu_S^2}{\nu_P^2} \right)^2} \) (\( \nu_S \) and \( \nu_P \) being the \( S \) and \( P \) wave speeds, respectively). A similar dependence of \( E_G \) on \( v_r \) was also reported by Day [1982] in his equation (11). Moreover, \( F(v_r) \) monotonically decreases for increasing \( v_r \) with maximum value for \( v_r = 0 \), where it equals 1/(1-\( \nu \)) or 1 in the case of mode II or mode III, respectively (\( \nu \) is the Poisson ratio).

[13] The dependence of \( E_G \) on \( u_{\text{tot}} \) contained in equation (7) roughly agrees with the empirical estimates of
Zhang et al. [2003], on the basis of the SW curves inferred from data of the 1999 Chi-Chi earthquake, as

\[ E_G = 0.35 \times 10^6 u_{tot}^{2.2}. \]  

(9)

[14] On the other hand, Abercrombie and Rice [2005] by fitting earthquake data for slip ranging from 0.2 mm to 0.2 m to their equation (5) found the slightly different relation

\[ E_G = 5.25 \times 10^6 u_{tot}^{1.28}. \]  

(10)

McGarr et al. [2004] in their equation (7), using a crack model, suggest that fracture energy is linearly related to slip, while laboratory experiments by Chambon et al. [2006, equation (10)] suggest a still lower scaling exponent (~0.6).

[15] From the nonspontaneous (i.e., with an a priori- assigned and constant \( v_s \)), 2D plane-strain, self-similar solution of Burridge [1973], Andrews [1976b] (p. 5685) obtains an expression for fracture energy density as

\[ E_G = \frac{\pi r}{4G} K(v_s) Q(v_s) \Delta \tau_d^2, \]  

(11)

where \( r \) is the distance of the rupture tip from the nucleation point (i.e., the length of the crack from the nucleation point), \( K(v_s) \) and \( Q(v_s) \) are dimensionless functions of the rupture velocity, and \( \Delta \tau_d = \tau_0 - \tau_f \) is the dynamic stress drop (\( \tau_0 \) denotes the initial shear stress). The square dependence of \( E_G \) on \( \Delta \tau_d \) postulated also by Kostrov [1964], recalls the proportionality of \( E_G \) to \( \Delta \tau_d^{1.85} \), contained in equation (6). The dependence of \( E_G \) on \( v_s \) in (11) can be made more explicit by considering equations (7) and (23) of Ida [1972], that give [see Andrews, 1976a, equation (23)]

\[ E_G = \frac{\pi r}{2G} B(v_s) \alpha S \Delta \tau_d^2. \]  

(12)

where \( \alpha S \) has been defined above and \( B(v_s) \) is a dimensionless, monotonic function of \( v_s \). Since \( B(v_s) \) can be approximated as \( 2/\pi \) [see Andrews, 1976a], equation (12) can be rewritten as

\[ E_G \propto \frac{r}{G} \alpha S \Delta \tau_d^2. \]  

(13)

This relation is equivalent to equation (11.24) of Aki and Richards [2002], derived in the case of a semi-infinite mode III shear model with cohesive force. A similar expression for \( E_G \),

\[ E_G = \frac{r}{\pi G} \frac{1}{\alpha} \Delta \tau_d^2, \]  

(14)

from equation (7) of Wong [1982], has been also used by Husseini et al. [1975] with \( \alpha = 1 \) in the case of a semi-infinite longitudinal shear crack.

[16] An important distinction between equation (7) and equations (12–14) is that the former is appropriate for a pulselike solution (in which the fault slip heals and the slip velocity has a finite duration), while the latter refer to a cracklike solution (where the slip does not spontaneously heal and the rupture continues to develop until the ends of the fault are reached or frictional heterogeneities are encountered).

[17] We finally mention that the kinematic models of Tinti et al. [2005] suggest that \( U_G \) scales with \( M_0 \) as

\[ U_G \propto M_0^p, \]  

(15)

where the exponent \( p \) is 1.18. This power law dependence, which links two quantities representative of the entire faulting episode, is in agreement with the fit of Abercrombie and Rice [2005] based upon observations arising from several real-world earthquakes; rewriting our equation (5) as

\[ U_G = A(E_G) \]  

with \( A \) being the average fracture energy density over the cracked area \( A \) and expressing \( E_G \) as in equation (5) of Abercrombie and Rice [2005], after simple algebra we have

\[ U_G \approx \frac{5.25 \times 10^6}{G^{4/3} A^{p+\frac{1}{3}}} M_0^q \]  

(16)

with a scaling exponent \( q = 1.28 \). (Note that we neglect both overshoot and undershoot phenomena, so that the quantity \( G' \) of Abercrombie and Rice [2005] is in fact our \( E_G \). Moreover, we express the average fault slip \( S \) in equation (5) of Abercrombie and Rice [2005] as a function of the scalar seismic moment \( S = M_0/G'(A) \) [e.g., Aki, 1967]). On the other hand, from equation (1) of Venkataraman and Kanamori [2004] we have

\[ U_G \approx C \frac{M_0^2}{2G A^{7/4}} \]  

(17)

where \( C \) is a dimensionless constant (\( C \approx 1.4 \)) and \( U_R \) is the (total) radiated energy (which is defined as the wave energy that would be transmitted to infinity if an earthquake occurred in an infinite, lossless medium) [Haskell, 1964]. Equations (15) to (17) suggest that fracture energy measured at the laboratory scale is several orders of magnitude smaller that that inferred for earthquakes [see also Chester et al., 2005].

3. Limitations of the Theoretical Predictions and Motivation of the Present Paper

[18] Equations (7), (11), (12), and (13) previously discussed have been derived within the framework of the LEM and by the identification of the limiting rupture speed of a propagating crack tip singularity. One important limitation of the above-mentioned equations we want to emphasize is that they have been (necessarily) obtained under the general assumption of simple 2D ruptures (or pulses). For example, equations (7) and (8) provide a relationship between \( E_G \) and \( v_s \) in the case of mode II ruptures only for \( v_s \leq v_R \) (from (7) and (8)) we have that \( E_G \) becomes negative for \( v_s > v_R \) where \( v_R \) indicates the Rayleigh speed and in the case of mode III ruptures only for \( v_s \leq v_S \) (from the definition of \( \alpha S \) we have from equations (7), (12), and (13) that \( E_G \) assumes complex values for \( v_s > v_S \). Indeed, Bhat et al. [2007] observe that for supershear slip pulses it is not possible to express \( E_G \) as a simple analytical expression of \( v_s \) (as in the subshock case); they numerically found (their Figure 14) that \( E_G \) initially increases for \( v_S < v_s \leq 1.3v_S \) and then decreases for \( 1.3v_S \leq v_s \leq v_R \). For steady state 2D shear cracks the velocity range between \( v_S \) and \( v_F \) is energetically inadmissible and therefore we cannot retrieve
any information in this velocity interval from the theoretical equations previously described.

[19] On the other hand, kinematic models contain only constraints on the average rupture velocity on a mathematical fault plane, ideally neglecting all possible small scale heterogeneities in geometrical path, such as branching, bending, and kinks, which have been introduced in dynamic models (e.g., E. M. Dunham et al., Earthquake ruptures with strongly rate–weakening friction and off–fault plasticity: 2. Nonplanar faults, submitted to Bulletin of the Seismological Society of America, 2010). More importantly, they cannot account for possible fluctuations of rupture velocity at periods shorter than those used to invert seismograms (limited frequency bandwidth limitation) and they finally suffer intrinsic limitations in representing the physics of earthquake rupture. Moreover, in all models of Tinti et al. [2005] $v_r$ is almost uniform and they do not give an estimate of the dependence of $E_G$ on $v_r$.

[20] As a natural consequence of the aforementioned limitations, in the present paper we mainly aim to explore whether fully dynamic, spontaneous models of earthquake ruptures developing on planar fault of finite extension indicate some specific relationships between fracture energy density and physical observables. In addition, we examine whether theoretical predictions based on some specific assumptions can be used also in more complex configurations. We will also investigate whether the behavior of fracture energy density is affected by the choice of the fault governing law and by the assumed spatial distribution of the initial shear stress on the fault surface.

[21] We finally remark that the calculations presented in this paper do not include inelastic deformations occurring near the rupture front. We also neglect the energy loss due to off-fault damage which increases the fracture energy density, but it can be adequately modeled by fracture energy on the fault [Andrews, 2005]. We do not consider zones of plastic deformation developing before the crack grows, nor viscous flow due to melting, so that there is no need to consider Elastic Plastic Fracture Mechanics (EPFM).

4. Rupture Simulations

[22] In this paper, we solve the fundamental elastodynamic equation, neglecting body forces, for a single, planar, strike-slip fault embedded in a perfectly elastic, isotropic half-space with free surface condition. The adopted fault

Figure 2. Geometry of the model. The imposed hypocenter is indicated by H and the light shaded plane indicates the fault $x_2 = x_2^f$, having aspect ratio $L^f/W^f$. The shaded box marks the portion of the computational domain where calculations are performed, due to the exploitation of the symmetry about H and about the fault plane.
Table 1. Model Discretization and Constitutive Parameters Adopted in This Study

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lamé constants, ( \lambda = G )</td>
<td>27 GPa</td>
</tr>
<tr>
<td>( S ) wave velocity, ( v_S )</td>
<td>3 km/s</td>
</tr>
<tr>
<td>( P ) wave velocity, ( v_P )</td>
<td>5.196 km/s</td>
</tr>
<tr>
<td>Cubic mass density, ( \rho )</td>
<td>3000 kg/m³</td>
</tr>
<tr>
<td>Fault length, ( L' )</td>
<td>12 km</td>
</tr>
<tr>
<td>Fault width, ( W' )</td>
<td>11.6 km</td>
</tr>
<tr>
<td>Spatial grid size, ( \Delta x_1 = \Delta x_2 = \Delta x_3 = \Delta x )</td>
<td>8 m³</td>
</tr>
<tr>
<td>Time step, ( \Delta t )</td>
<td>4.44 × 10⁻⁴ s</td>
</tr>
<tr>
<td>Courant-Friedrichs-Lewy (CFL) ratio, ( \omega \text{CFL} = v_S \Delta t/\Delta x )</td>
<td>0.1665</td>
</tr>
</tbody>
</table>

| Coordinates of the hypocenter \( H = (x^H_1, x^H_2) \) | \((5.992,7)\) km |
| Domain boundary conditions | \( x_1 = 0: \text{ABC}^a; x^H_1: \text{symmetry}^c \) |
| | \( x_2 = 0: \text{ABC}^b; x^H_2: \text{symmetry}^e \) |
| | \( x_3 = 0: \text{Free surface}; = x^H_3: \text{ABC}^b \) |

Fault Constitutive Parameters

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Initial rake angle, ( \varphi_0 )</td>
<td>120 MPa</td>
</tr>
<tr>
<td>Effective normal stress, ( \sigma_{\text{eff}} )</td>
<td>70.51572 MPa³</td>
</tr>
</tbody>
</table>

Slip-Weakening Law (Equation (18))

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Magnitude of the initial shear stress, ( \tau_0 )</td>
<td>70.51572 MPa³</td>
</tr>
<tr>
<td>Static level of friction coefficient, ( \mu_r )</td>
<td>0.73167 (( \Rightarrow \tau_0 = 87.80) MPa)³</td>
</tr>
<tr>
<td>Kinetic level of friction coefficient, ( \mu_k )</td>
<td>0.54333 (( \Rightarrow \tau_r = 65.20) MPa)³</td>
</tr>
<tr>
<td>Dynamic stress drop, ( \Delta \sigma_d = \tau_0 - \tau_f )</td>
<td>5.32 MPa</td>
</tr>
<tr>
<td>Breakdown stress drop, ( \Delta \tau_b = \tau_0 - \tau_f )</td>
<td>22.60 MPa</td>
</tr>
<tr>
<td>Strength parameter ( S = (\tau_0 - \tau_0)(\tau_0 - \tau_f) )</td>
<td>3.25</td>
</tr>
<tr>
<td>Characteristic slip-weakening distance, ( d_0 )</td>
<td>0.05 m³</td>
</tr>
</tbody>
</table>

Ruina-Dieterich Law (Equation (19))

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Logarithmic direct effect parameter, ( a )</td>
<td>0.016</td>
</tr>
<tr>
<td>Evolution effect parameter, ( b )</td>
<td>0.020</td>
</tr>
<tr>
<td>Scale length for state variable evolution, ( L )</td>
<td>0.02 m</td>
</tr>
<tr>
<td>Reference value of friction coefficient at low slip rates, ( \mu^* )</td>
<td>0.56</td>
</tr>
<tr>
<td>Initial sliding velocity, ( v_0 )</td>
<td>1 × 10⁻³ m/s</td>
</tr>
<tr>
<td>Magnitude of the initial shear stress, ( \tau_0 )</td>
<td>( \mu^*(v_0) ) ( \sigma_{\text{eff}}^a = 70.51572 )³</td>
</tr>
</tbody>
</table>

Ruina-Dieterich Law With Flash Heating of Asperity Contacts (Equation (20))

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Reference value of friction coefficient at high slip rates, ( \mu_b )</td>
<td>0.13</td>
</tr>
<tr>
<td>Initial sliding velocity, ( v_0 )</td>
<td>1 × 10⁻³ m/s</td>
</tr>
<tr>
<td>Magnitude of the initial shear stress, ( \tau_0 )</td>
<td>( \mu^*(v_0) ) ( \sigma_{\text{eff}}^a = 70.51572 )³</td>
</tr>
</tbody>
</table>

*Fine spatiotemporal discretization guarantees a proper resolution of the breakdown zone (see Bizzarri, [2009a] for numerical details) for all of the considered governing models. This allows the analysis of ruptures up to a frequency \( f_{\text{breakdown}} = v_S/(6\Delta x) = 62.5 \) Hz and what is more, it guarantees not less than 40 points within the breakdown zone (on average, for all considered numerical simulations), that in turn ensures a stable determination of fracture energy density in each fault node.

*Absorbing boundary conditions described by Bizzarri and Spudich [2008, Appendix A].

*Symmetries about the strike location of the hypocenter \( (x_1 - x^H_1) \) and about the fault \( (x_3 - x^H_3) \) are exploited as described by Bizzarri [2009a].

*For sake of simplicity the initial shear traction vector is aligned along \( x_3 \).

*This value has been chosen to have the same initial shear stress for all governing models in case of homogeneous conditions.

*These values correspond to the average values of \( \tau_0^a, \tau_r^a \) and \( d_0^b \) of the homogeneous RD simulation.

*The \( \mu^a \) denotes the steady state value of the friction coefficient, realized when \( d\sigma/dt = 0 \) in the evolution equations of models (19) and (20).

Figure 3. Solution for a synthetic subshear earthquake, obeying the RD law (equation (19)), and for a homogeneous distribution of the initial shear stress. Distribution on the fault plane at the final time level of the numerical simulation of (a) cumulative fault slip \( u_{\text{tot}} \), (b) fault slip velocity \( v_f \), (c) fault traction \( \tau_f \), (d) rupture velocity \( v_r \), and (e) \( E_{\text{fracture}} \). The insets in Figures 3a–3c report the behavior of \( u_{\text{tot}}, v_f, \) and \( \tau_f \), respectively, as functions of the strike coordinate, at the hypocentral depth. Rupture velocities are calculated as in equation (12) of Bizzarri and Spudich [2008] using \( v_f(x_1, x_3) = 1/\left| \nabla_e (x_1, x_3) \right| \), where \( t_r \) is the rupture time, defined as the instant of time at which the fault slip velocity exceeds a threshold value \( v_f \) assumed to be 0.01 m/s. In Figures 3a, 3b, 3d, and 3e purple color denotes unbroken part of the fault plane. Note that due to symmetry exploitation only one-half of the fault along the strike direction is reported on the plots (the same holds for Figures 10a, 10c, 10d, 11).
The problem is solved numerically, by employing the three-dimension (3D), second order accurate, OpenMP-parallelized, finite difference, conventional grid code described by Bizzarri and Cocco [2005]. The absorbing boundary conditions described in Bizzarri and Spudich [2008] are adopted in order to reduce spurious reflections from the borders of the computational domain and the existing symmetries are exploited to reduce...
computational times and storage requirements, as discussed in detail by Bizzarri [2009a]. The rupture starts from the hypocenter $H$ and expands bilaterally in a spontaneous fashion and dynamically; the slip is purely tangential, so that no opening or interpenetration of material is allowed. Within an initialization patch $I_{nucl}$ surrounding $H$, the earthquake nucleation is realized by initially forcing the rupture to develop with a constant speed in the case of the SW law. For the RS law, on the contrary, the nucleation is obtained by decreasing at $t = 0$ the value of the state variable, by setting a value smaller than the steady state attained outside $I_{nucl}$. Additional numerical details are presented by Bizzarri [2010a] and Bizzarri [2009a].

The fault boundary condition on the frictional interface is represented by the constitutive law (in this study we consider a wide range of governing models): under the linear SW law,

$$\tau = \left\{ \begin{array}{ll} \tau_a - (\tau_a - \tau_f) \frac{u}{d_0}, & u < d_0 \\ \tau_f, & u \geq d_0 \end{array} \right.,$$  

(Ida, 1972), under the Ruina-Dieterich (RD) form of the RS law,

$$\tau = \left[ \mu_* + a \ln \left( \frac{v}{v_*} \right) + \Theta \right] \sigma_{\theta}^{\text{eff}}$$

$$\frac{d}{dt} \Theta = -\frac{v}{L} \left[ b \ln \left( \frac{v}{v_*} \right) + \Theta \right]$$

[Bizzarri, 2009a, and references therein] ($\Theta$ is the dimensionless state variable, $a$ is a constitutive parameter, while $\mu_*$ and $v_*$ are reference values for friction coefficient and sliding velocity, respectively), and under the RD law with the incorporation of the phenomenon of the flash heating (FH) of microscopic asperity contacts,

$$\tau = \left[ \mu_* + a \ln \left( \frac{v}{v_*} \right) + \Theta \right] \sigma_{\theta}^{\text{eff}}$$

$$\frac{d}{dt} \Theta = -\frac{v}{L} \left[ b \ln \left( \frac{v}{v_*} \right) + \Theta \right] + \left( 1 - \frac{\mu_{fh}}{\mu_*} \right) \left( a \ln \left( \frac{v}{v_*} \right) + \mu_* - \mu_{fh} \right)$$

[Bizzarri, 2009a, and references therein], where $v_{fh}$ is the cutoff velocity above which FH operates (for $v \leq v_{fh}$ the evolution equation for $\Theta$ is that of the classical RD model (19)) and $\mu_{fh}$ is the reference value for friction coefficient at high slip velocities.

[24] The reference parameters adopted in this study (see Table 1) refer to a typical, subshear crustal earthquake. In the next two sections we will assume on the fault a homo-

**Figure 4.** Relations between fracture energy density and physical observables for the simulation reported in Figure 3. With red circles we report (a) $E_G$ as a function of $v_r$, (b) $E_G$ as a function of $u_{tot}$ and (c) $E_G$ as a function of $D_{t,b}$. In orange are reported the values of $E_G$ as obtained from theoretical predictions (equations (13), (7), and (11) for Figures 4a, 4b, and 4c, respectively). The dashed blue curve indicates the value of $E_G$ for corresponding homogeneous SW model (equation (18)), as obtained from equation (2).
geneous initial shear stress, while in section 7 we will consider heterogeneous distributions.

5. Results for Homogeneous Configurations: Subshear Synthetic Earthquakes

[25] In homogeneous conditions, the linear SW model, equation (18), prescribes a constant fracture energy density over the whole fault; $E_G$ is simply expressed by equation (2). On the contrary, within the framework of RS friction laws $E_G$ can be spatially variable, due to the variability of shear traction on the fault, even for homogeneous $\tau_0$.

[26] We report in Figure 3 the spatial distribution on the fault plane of the relevant quantities of our problem at the end of a numerical experiment. They pertain to a synthetic
earthquake of moderate size ($M_0 = 4.25 \times 10^{17}$ Nm, corresponding to $M_w = (\log(M_0) - 9)/1.5 = 5.8$) obeying the RD law, equation (19). In Figure 4 we plot, with red circles, the behavior of $E_G$, represented separately as a function of $v_r$, $u_{tot}$ and $\Delta \tau_{\text{d}}$, resulting for this synthetic event. On the basis of our fine spatial discretization (see Table 1), in each synthetic event about one million points were considered for the analysis. We have performed a zero-offset spatial correlation analysis, in that we have considered the values of different quantities attained in the same fault node. We will discuss in the appendix the effects of a nonzero-offset correlation analysis, where the quantities are compared in different points of the rupture plane (i.e., we introduce some spatial offset).

From Figure 4a we can see that the dependence of $E_G$ on $v_r$ is roughly described by equation (13) (orange dots); the fluctuations for low $v_r$ ($v_r < 500$ m/s) refer to the nucleation phase, within the initialization patch $I_{\text{nuc}}$. Note that equation (13) in Figure 4a is plotted for the different values of $\Delta \tau_{\text{d}}$ realized in the different fault nodes. On the contrary, from Figure 4b we have that the proportionality of $E_G$ to $u_{\text{tot}}$ theoretically predicted for pulses by equation (7) (orange dots) is markedly far from describing the numerical results (Figure 4b, red dots); this result is not surprising and we will discuss in the next section the reason for such a strong disagreement with the theoretical prediction. Finally, Figure 4c shows that the dependence of $E_G$ on $\Delta \tau_{\text{d}}$ is somehow proportional to $\Delta \tau_{\text{d}}^2$, as expected from equation (11).

In Figure 5 we report the 3D scatter plots of the fracture energy density as a function of the two independent variables appearing in equations (7) and (11). This kind of figure is complementary with respect to Figure 4; for example, by considering the data points in Figure 5a, we have that the projection on the $E_G - v_r$ plane gives Figure 4a, while the projection on the $E_G - u_{\text{tot}}$ plane gives Figure 4b. (The same holds for Figure 5b.)

6. Homogeneous, Supershear Synthetic Earthquakes

In this section we will consider three cases, representative of supershear seismic events, where the maximum speed asymptotically reaches the $P$ wave velocity. Even though most natural earthquakes have subshear rupture velocities, there is increasing interest in supershear earthquakes, because they have some important and distinct features [Bizzarri and Spudich, 2008; Dunham and Bhat, 2008; Bizzarri et al., 2010].

One numerical simulation refers to the FH governing model (equation (20)), in which, if the local temperature of a microscopic asperity contact reaches a temperature at which thermally activated defects become highly mobile, then the contact will weaken. The inclusion of FH causes the transition to the supershear regime [Bizzarri, 2009a] for a rheology (that of Table 1) which would produce a subshear propagation in the absence of FH (see previous section). The results are reported in Figures 6 and 7. In the other two cases, the RD law (equation (19)) is adopted, but we change the value of governing parameters $a$, in order to increase the degree of instability of the fault and its propensity to accelerate up to supershear speeds [see also Bizzarri et al., 2001].

Figure 6. The same as in Figure 4, but now in the case of a supershear rupture obeying the FH governing law (equation (20)).
The results are displayed in Figures 8 and 9. Figure 8 compares the results obtained using $a = 0.012$ (full circles) and $a = 0.010$ and $b = 0.022$ (open circles). While the value of the initial shear stress in the FH case is the same as that adopted in the simulations discussed in the previous section, the different values of parameters $a$ and $b$ change the magnitude of $\tau_0$ in the two RD numerical experiments.

It is apparent from Figures 6a and 8a that the relationship between $E_G$ and $v_r$ becomes complicated in the case of supershear ruptures. In the FH case (Figure 6a) there is no a clear trend of $E_G$ for increasing $v_r$, since the data are very sparse; they tend to group around two values of $v_r$, approximately at 2.5 km/s and 4.2 km/s (visible also in Figures 7a and 7b). In the RD cases (Figure 8a) it is possible

Figure 7. The same as in Figure 5, but now in the case of the FH model reported in Figure 6.

The results are displayed in Figures 8 and 9. Figure 8 compares the results obtained using $a = 0.012$ (full circles) and $a = 0.010$ and $b = 0.022$ (open circles). While the value of the initial shear stress in the FH case is the same as that adopted in the simulations discussed in the previous section, the different values of parameters $a$ and $b$ change the magnitude of $\tau_0$ in the two RD numerical experiments.

[31] It is apparent from Figures 6a and 8a that the relationship between $E_G$ and $v_r$ becomes complicated in the case of supershear ruptures. In the FH case (Figure 6a) there is no a clear trend of $E_G$ for increasing $v_r$, since the data are very sparse; they tend to group around two values of $v_r$, approximately at 2.5 km/s and 4.2 km/s (visible also in Figures 7a and 7b). In the RD cases (Figure 8a) it is possible
to roughly envisage the behavior found by Bhat et al. [2007] in the case of supershear slip pulses. In our two spontaneous, dynamic ruptures with RD law there are large fluctuations of data and it is not simple to identify the value of rupture velocity which maximizes the fracture energy density; a rough estimate is that it is slightly smaller than $v_s$.

A essential difference emerges from the comparison of the $E_G$ versus $u_{tot}$ curves (Figures 6b and 7a versus Figures 8b and 9a). We can see that the general agreement with equation (7) is quite good in the case of FH simulation (Figures 6b and 7a), but from the RD simulation shown in Figures 8b and 9a we have that $E_G$ increases roughly in a linear fashion for increasing slip. We emphasize that in the FH case, for the adopted parameters, we have a pulselike solution, while in the two RD cases we have cracklike solutions (in these cases $u_{tot}$ has to be interpreted as the slip at the end of the numerical experiments). Therefore, our fully dynamic simulations confirm that the prediction by Rice et al. [2005] is not appropriate in the case of sustained cracklike ruptures (see also Figures 4b and 5a). This result, which is also corroborated by the kinematic findings of Tinti et al. [2005], holds for other FH models where governing parameters produce slip pulses.

In Figures 6c and 8c we plot the $E_G$ versus $\Delta \sigma_d$ curves. In the case of a sustained supershear slip pulse (Figures 6c and 7b) the value of $\tau_{eq}$ is substantially spatially constant. Considering that $\tau_0$ is homogeneous, this causes the dynamic stress drop to be basically constant over the whole fault ($\Delta \sigma_d \approx 55$ MPa). This is apparent from Figures 6c and 7b, where we can see that the smaller values of $\Delta \sigma_d$ correspond to very low values of the fracture energy density, attained basically within $\nu_{nuc}$. The fluctuations of $E_G$ for $\Delta \sigma_d \approx 55$ MPa are due to the variations in traction evolution for slip below $d_0^e$ (recall equation (1)). In Figures 8c and 9b we can see that in these supershear ruptures $E_G$ roughly increases linearly for increasing dynamic stress drop. In general, we can conclude that in both supershear cases, FH pulse and RD ruptures, the behavior predicted by equation (11) is not satisfied.

7. Do Heterogeneities Play a Role?

Frictional sliding is usually a very irregular process, due to inhomogeneous conditions on the sliding surfaces [Broberg, 1978]. To account for this, following Bizzarri et al. [2010], in the present section we assume that the magnitude of the initial shear stress has a $k^{-1}$ behavior at high wave numbers. Namely, $\tau_0$ has a power spectral density (PSD) which is

$$\sqrt{P(k)} = \frac{1}{\left(1 + \left(\frac{k}{k_c}\right)^2\right)^{1+D}}, \quad (21)$$

where $k = ||\mathbf{k}|| = \sqrt{k_1^2 + k_3^2}$, with $k_1$ and $k_3$ being the horizontal (along strike) and vertical (along depth) wave numbers, respectively. In (21) $k_c$ is the radial wave number corresponding to the correlation length $\chi_c$ ($k_c = 2\pi/L_c$), and $D$ is the dimensionless Hurst exponent [see Mai and Beroza, 2002]. In equation (21) we choose $D = 0$, which corresponds in the static limit to the well-known “$k$-square”
Figure 9. The same as in Figure 5, but now for the RD model reported in Figure 8, with $a = 0.012$ and $b = 0.020$.

Figure 10. (a) Magnitude of the initial shear stress adopted in heterogeneous simulations; the fluctuations with respect to the reference value of $\tau_0$ listed in Table 1 (green color) follow the desired power spectral density (see section 7 for details). (b) Resulting Fourier amplitude spectra as a function of the radial wave number $k$; the dashed red curve emphasizes the required $k^{-1}$ behavior. (c) Distribution of rupture velocity on the fault plane in the case of classical RD law. (d) Same as Figure 10c, but in the case of the SW law.
Figure 10
model [Herrero and Bernard, 1994] of slip at high wave numbers, corroborated by several inversions of ground motion data. We also set \( L_c = 1000 \text{ km} \) in order to have a power law spectrum of the initial shear stress for all modelled wavelengths.

[35] The distribution of the initial shear stress adopted in the heterogeneous numerical experiments is reported in Figure 10a, where the green color identifies the reference value of \( \sigma_0 \), reported in Table 1. The resulting one dimension (1D) spectrum is plotted in Figure 10b, from which we can see that its PSD behaves like \( k^{-3} \), as desired (dashed red curve); the RMS of such a distribution is \( 5.52 \text{ MPa} \). We apply the heterogeneous stress of Figure 10a in two numerical experiments, one with the classical RD law (equation (19)) and the other one with the SW law (equation (18)). In the latter case, the heterogeneities in \( \sigma_0 \) will affect rupture times, rupture velocity, and total cumulative slip, but when \( \tau_u \), \( \tau_r \), and \( d_0 \) are spatially homogeneous, the resulting \( E_G \) will be constant (see equation (2)), as in a homogeneous simulation. Therefore, to allow for a variable \( E_G \) in the SW case, we also add heterogeneities in \( \tau_u \), which is spatially variable, such that the strength excess \( \tau_u - \tau_0 \) is constant (and equal to the reference value of the homogeneous configuration reported in Table 1). The other constitutive parameters are those tabulated in Table 1.

[36] The two resulting models behave very differently; this is clear from the comparison of Figures 10c and 10d, where we report the distribution of \( r \) on the fault plane. On the basis of the fluctuations of \( \sigma_0 \) in the RD case there are multiple points on the fault where the rupture starts to propagate dynamically. This behavior is similar to that obtained by Bizzarri and Spudich [2008, Figure 11]; the multiple ruptures interact with one another and insert a further complication into the model, other than the imposed heterogeneity of \( \sigma_0 \). On the contrary, the SW simulation exhibits a more usual behavior; rupture initiates within the nucleation patch and then propagates dynamically. In both cases, the presence of heterogeneities complicates the evolution of the rupture front and furthermore it causes, especially in the SW model, local acceleration to supershear rupture speeds. In these two models, both subshear and supershear rupture patches exist. Correspondently, the distributions on the fault plane of \( u_{\text{tot}} \) and \( E_G \) are quite different between two simulations (compare Figure 11a with Figure 11b and Figure 11c with Figure 11d).

[37] We report in Figure 12 the resulting behavior of \( E_G \) as a function of rupture velocity (Figure 12a), total cumulative fault slip (Figure 12b), and dynamic stress drop (Figure 12c). We can clearly see that \( E_G \) exhibits large fluctuations due to frictional heterogeneities. In this case it is hard to find a clear and well-defined trend of \( E_G \) as a function of \( r \). We can also see that in the RD model \( E_G \) increases roughly linearly for increasing \( u_{\text{tot}} \) (Figure 12b, red symbols), as previously observed in the homogeneous sustained supershear RD simulations (Figures 8b and 9a). In the SW case, \( E_G \) does not have a strong dependence on \( u_{\text{tot}} \) (Figure 12b, blue symbols); it is oscillating above and below a constant value, slightly greater to the reference value of fracture energy density pertaining to the homogeneous configurations (equation (2)). Large fluctuations are also present in the \( E_G \) versus \( \Delta \tau_d \) curve (Figure 12c), which indicate a linear increase of \( E_G \) for increasing \( \Delta \tau_d \).

8. Discussion

[38] In previous sections, we have reported the results of different numerical simulations that are representative of different fault governing laws, rupture regimes, and initial conditions. We have considered a grid-by-grid analysis to show the behavior of fracture energy density as a function of physical observables. In this section, we summarize results of a different kind of analysis, which is performed by considering event by event. In particular, we take into account the whole ensemble of simulations, comprised of 40 numerical experiments of fully dynamic and spontaneous seismic ruptures that cover a broad magnitude range (\( M_0 \) from \( 4.25 \times 10^{17} \text{ Nm} \) up to \( 1.07 \times 10^{19} \text{ Nm} \); corresponding to \( M_w \) from 5.8 to 6.7). For each synthetic event we consider the spatial averages, over the fault nodes experiencing the rupture, of \( E_G \), \( r \), \( u_{\text{tot}} \), and \( \Delta \tau_d \) (in the following we denote them with symbols \( \langle E_G \rangle \), \( \langle r \rangle \), \( \langle u_{\text{tot}} \rangle \), and \( \langle \Delta \tau_d \rangle \), respectively.) The results are shown in Figure 13. First of all, we emphasize that it is impossible to compare data against a single curve, exactly representing the theoretical predictions discussed in section 2, because they contain quantities that cannot be spatially averaged. For example, in equations (11) and (13) the quantity \( r \) denotes the distance of a rupturing fault node from the hypocenter \( H \), a quantity which cannot spatially averaged. On the other hand, \( t_{\text{pulse}} \) is equation (7) depends on the specific rupture we consider; thus, that multiplier factor is different from one event to another.

[39] A first observation coming from Figure 13 is that the results are quite sparse; this is not surprising, given the broad range of input parameters. On the other hand, the values estimated by kinematic inversion for different real earthquakes [Tinti et al., 2005; see also Abercrombie and Rice, 2005], represented by shaded triangles in Figure 13, also appear to be quite sparse.

[40] From Figure 13a we can see that, for subshear ruptures (open symbols), \( E_G \) behaves roughly like \( \sqrt{1 - \left( \frac{r}{r_s} \right)^2} \) (black curve in Figure 13a), as previously observed (Figure 4a). On the contrary, for supershear synthetic events (full symbols), data appear to be clustered around a value of \( \langle r \rangle \) nearly equal to Eshelby’s speed (\( v_k = \sqrt{2} v_b \)), but there is a large variability of \( \langle E_G \rangle \). In this rupture regime it is difficult to find a specific behavior of fracture energy density as a function of rupture velocity, in agreement with results previously discussed (see Figures 6a, 7b, 8, and 9b). It is apparent from Figure 13b that our results seem to agree better as a linear fit with \( (u_{\text{tot}}) \), as suggested by McGarr et al. [2004] (green curve in Figure 13b). This also confirms what we have observed in Figures 8b and 9a for a cracklike rupture. From Figure 13c we can see that \( \langle E_G \rangle \) roughly goes like \( \langle \Delta \tau_d \rangle^\alpha \), where \( \alpha \) is in between 1 and 2, as also found above (see Figures 4b and 12c). The only exception is represented by the sustained supershear FH pulses (group of pink diamonds on the right), that have extremely high dynamic stress drop (55 MPa for the case reported in Figures 6c and 7b).

[41] We have also checked if two parameters quantifying a faulting episode as a whole, the total fracture energy (\( U_G \); see equation (5)) and the scalar seismic moment (\( M_0 \), cor-
Figure 11. Distribution on the fault plane of (top) $\mu_{\text{tot}}$ and (bottom) $E_G$ pertaining to the heterogeneous rupture of (left) Figure 10c and (right) Figure 10d.
relate. In Figure 14 we report the values of $U_G$ and $M_0$ for the whole ensemble of synthetic earthquakes, obeying different constitutive laws and for homogeneous (squares) and heterogeneous (circles) conditions. We notice that these values refer to the rupture developed over the whole fault of length $L' = 12$ km; since the rupture is bilateral and symmetric with respect to the hypocenter we simply double the values obtained for the fault considered for computations, which extends only 6 km. In Figure 14 we have also superimposed, as shaded triangles, the values estimated from kinematic inversions by Tinti et al. [2005] for several real events. We can clearly see that our dynamic models exhibit a power law relationship between total fracture energy and scalar seismic moment, with a slope that depends very weakly on the inclusion of the fault points within the nucleation patch. In fact, full symbols (for which $U_G$ and $M_0$ are determined for all fault points) and open symbols (for which nodes within $I_{nuc}$ are not considered in the calculation of $U_G$ and $M_0$) indicate slopes of 1.13 and 1.14, respectively (solid and dashed red curves in Figure 14, respectively). This power law exponent is in agreement with the value of 1.18 estimated in the kinematic models of Tinti et al. [2005] (shaded curve). For comparison we also plot in Figure 14 the power law found by Abercrombie and Rice [2005] (see equation (15)), having an exponent $q = 1.28$ (black curve).

9. Conclusions

[42] In addition to the amount of dissipation occurring during healing process [Broberg, 1978], the fracture energy density ($E_G$) is one of the key parameters in the physics of the earthquake source [e.g., Kostrov, 1964]. Compared to other source parameters, $E_G$ has been proved to be more stably estimated from kinematic rupture models inferred from waveform inversions of strong motion data [Guatteri and Spudich, 2000] and has an important influence in discriminating between melting and nonmelting regimes [Bizzarri, 2010b]. The basic objective of the present paper is to see if $E_G$ exhibits some specific dependencies on the most prominent physical quantities (observables or dynamic variables), such as the rupture velocity ($v_r$), the total cumulative fault slip ($u_{tot}$) and the dynamic stress drop ($\Delta \tau_d$). This has been done for scenarios more realistic and more complex than (necessarily) simplified models for which some prediction has been derived theoretically.

[43] In general, to accurately predict ground motions by a kinematic numerical model and for simulation-based seismic hazard analysis, it is necessary not only to know the spatial distribution of the source parameters, but also to know the correlations among them [e.g., Song et al., 2009; Schmedes et al., 2010].

[44] To explore these relationships we have performed 40 numerical simulations of fully dynamic and spontane-

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**Figure 12.** Results for heterogeneous configurations of Figures 10 and 11; red symbols refer to RD model and blue symbols refer to the SW model. (a) $E_G$ as a function of $v_r$. (b) $E_G$ as a function of $u_{tot}$. (c) $E_G$ as a function of $\Delta \tau_d$. The dashed blue curve indicates the reference value of $E_G$ for the homogeneous SW model, as given by equation (2).
ous seismic ruptures that cover a wide size range ($M_0$ from $4.25 \times 10^{17}$ Nm to $1.07 \times 10^{19}$ Nm, corresponding to $M_w$ from 5.8 to 6.7) and span a broad range of relevant situations. The synthetic earthquakes are 3D ruptures (with rake rotation allowed and a possible continuous transition from sub to supershear regimes) and develop on a planar fault of finite size, which is governed by different constitutive laws, the linear slip-weakening (SW) function (equation (18)), the Ruina-Dieterich (RD) form of rate- and state-dependent laws (equation (19)), and the RD law...
would not be a precisely defined concept. 

\[ E = \frac{1}{2} v_i - u_E \]

is the radiated energy. 

Results from the event as a function of (a) \( R_v \) [2005, Table 5, and references therein], by using the relation 

\[ E = \frac{1}{2} G_i h_i \]

are indicated in the legends. In Figure 13c the triangles surrounded by a circle are spatially averaged, for each wave speed and in the generally is a non-can be regarded as meso-

\[ E = \frac{1}{2} G_i h_i \]

Relation between total fracture energy 

where points within the nucleation patch are considered; open symbols denote the estimates of \( U_G \) obtained neglecting points within \( I_{nucl} \). Solid and dashed red curves are the fits on synthetic data (as including and neglecting points within \( I_{nucl} \) respectively). For comparison we report the power law by Tinti et al. [2005] (shaded curve, with exponent of \( M_0 \) equal to 1.18) and that found by Abercrombie and Rice [2005] (black curve, with exponent of \( M_0 \) equal to 1.28).

Figure 13. Results from the event-by-event analysis; the quantities \( E_G, v_r, u_{tot} \), and \( \Delta \tau_d \) are spatially averaged, for each event, over the points that fail. Red symbols refer to numerical experiments in which classical RD law is adopted, blue symbols refer to SW models, and pink symbols refer to FH simulations. Circles refer to models with heterogeneous conditions; diamonds indicate pulselike solutions. Open and full symbols denote sub- and supershear ruptures, respectively. For comparison we also add, as shaded triangles, the estimates obtained from kinematic inferences for some real earthquakes by Tinti et al. [2005] (reader can refer to that work for references about source models). \( \langle E_G \rangle \) as a function of (a) \( \langle v_r \rangle \), (b) \( \langle u_{tot} \rangle \), and (c) \( \langle \Delta \tau_d \rangle \). In Figure 13a the black curve reproduces the behavior \( \sim \sqrt{1 - \left( \frac{v_r^2}{v_s^2} \right)} \). In Figure 13b we report different fits from the literature (various exponents of \( u_{tot} \) are indicated in the legends). In Figure 13c the triangles surrounded by a circle are extracted from the works by Abercrombie and Rice [2005, Table 5, and references therein], by using the relation 

\[ E_G = (1/2\pi R^2)(\Delta \tau_d M_0/G) - E_S \]

where \( R \) is the source radius and \( E_S \) is the radiated energy.

Figure 14. Relation between total fracture energy \( U_G \), calculated from equation (5), and scalar seismic moment \( M_0 \) calculated as described by Bizzarri and Belardinelli [2008]. Full symbols denote the calculations of \( U_G \) where points within the nucleation patch are considered; open symbols denote the estimates of \( U_G \) obtained neglecting points within \( I_{nucl} \). Solid and dashed red curves are the fits on synthetic data (as including and neglecting points within \( I_{nucl} \) respectively). For comparison we report the power law by Tinti et al. [2005] (shaded curve, with exponent of \( M_0 \) equal to 1.18) and that found by Abercrombie and Rice [2005] (black curve, with exponent of \( M_0 \) equal to 1.28).

In the literature different analytical relations have been reported, but they intrinsically suffer some limitations in basic assumptions (namely, they have been derived for rupture velocities not greater than \( S \) wave speed and in the case of simple 2D ruptures). On the other hand, presently available kinematic inversions of strong motion data do not give clear indication of the dependences of \( E_G \) on observables; thus, dynamic models can help us in filling this gap. Moreover, it is well known that fracture energy density from laboratory experiments are 3–5 orders of magnitude smaller than that inferred for earthquakes [e.g., Rudnicki, 1980], making it difficult to extrapolate laboratory results to natural conditions. As a consequence, physics-based earthquake source models represent a powerful tool to explore more general, and potentially more realistic, physical scenarios.

We can summarize the results we obtain from our numerical experiments as follows.

1. As is well-known, in homogeneous conditions the linear SW law gives a constant value of \( E_G \), which is uniform on the whole fault plane and is independent on physical observables. For a single event, the comparison with the theoretical predictions is inherently impossible in this case; for instance, \( E_G \) generally is a non-unique function of \( v_r \). On accounting for flash heating (FH) of asperity contacts (equation (20)). We have considered both homogeneous and heterogeneous distributions of the initial shear stress on the fault, and both subshear and supershear rupture events. Overall, by taking into account all the ruptures in our synthetic catalog, we have examined about 44 million fault points.
the other hand, by considering different events, with homogeneous constitutive parameters, $E_G$ can also be a non-unique function of $\Delta \tau_d$ (for fixed values of $\tau_m$, $\tau_f$ and $d_0$, different values of $\tau_0$ give different $\Delta \tau_d$ corresponding to the same value of $E_G$).

2. Subshear, homogeneous ruptures governed by RD law show a general agreement with the theoretical prediction (equation (13)) of $E_G \propto \sqrt{1 - (v_i^2/v_S^2)}$ (see Figures 4a and 13a).

3. On the contrary, for ruptures that accelerate up to supershear speeds it is extremely difficult to infer a clear dependence of fracture energy density on rupture speed (see Figures 6a, 8a, and 13b). Homogeneous simulations with RD law roughly indicate that $E_G$ seems to be a concave function of $v_r$ (see Figure 8a), in general agreement with the results obtained by Bhat et al. [2007] in 2D. However, the introduction of frictional heterogeneities further complicates this behavior and add some fluctuations to $E_G$ so that there is no an apparent dependence of $E_G$ on $v_r$ (see Figure 12a).

The existence of a relationship between $E_G$ and $v_r$ is intriguing and several physical motivations have been proposed to justify it (see Sharon et al. [1996] for a discussion), but we want to emphasize here that, especially for small earthquakes, there are some limitations in making estimates of the local rupture velocity of a real-world earthquake from the inversion of seismic waves. Moreover, the results can be non-unique. On the other hand, by considering a purely mode II crack growing on a granite sample, Ohnaka et al. [1987, equation (5)] found an empirical relation relating the peak slip velocity on the rupture plane, $v_{peak}$, $v_r$, and $\Delta \tau_d$,

\[ v_r \cong \frac{v_{peak} G}{\Delta \tau_d}, \quad (22) \]

which gives (once substituted into equation (7) and equations (11–13)) an implicit dependence of $E_{G}$ on $v_{peak}$. Equation (22) makes sense, since particle velocity and ground motions are determined by the rupture speed and this reinforces the fact that the fracture energy density is arguably one of the most important physical property in the earthquake source models. A strong correlation between $v_{peak}$ and $v_r$ has been also confirmed by the dynamic subshear numerical simulations of Schmedes et al. [2008].

4. On average, the spatial distributions on the fault surface of fracture energy density are correlated with the corresponding slip distributions (see Figure 11); high slip patches correspond to high $E_G$, in general agreement with the kinematic findings of Tinti et al. [2005]. This correlation is primarily due to the correlation of characteristic slip-weakening distance with slip, but also to the correlation of $\Delta \tau_d$ with $u_{tot}$. More specifically, our results in 3D confirm that slip pulses noticeably exhibit a behavior like $E_G \propto u_{tot}^2$ (Figures 6b and 7a), as predicted by the theory for 2D steady pulses (see equation (7)). On the contrary, in the cases of cracklike solutions, both sub- and supershear and both homogeneous and heterogeneous, this behavior is not confirmed by spontaneous, dynamic rupture models (see Figures 4b, 8a, and 12b). Our results (see Figure 13b) are in better agreement with the seismological inferences of McGarr et al. [2004] and results of dynamic rupture models of Mai et al. [2006, equation (5)], from which it emerges as a scaling exponent roughly equal to 1.

5. The proportionality between $E_G$ and $\Delta \tau_d$ expected from the theoretical predictions (see equations (11–13)) is somewhat verified in homogeneous, subshear ruptures with RD law (Figures 4c and 5b). On the contrary, in the cases of supershear rupture obeying the RD law (Figures 8c and 9b), and heterogeneous events (Figure 12c) our numerical experiments roughly suggest $E_G \propto \Delta \tau_d$ (see also Figure 13c). We want to remark that in the present simulations we neglected variability with depth of the effective normal stress, which would introduce a depth-dependence of stress drop.

6. Our spontaneous rupture models indicate that the total fracture energy ($U_G$, equation (5)) and the scalar seismic moment ($M_0$) correlate, with a power law dependence with an exponent equal to 1.13 (Figure 14), according to previous studies [e.g., Tinti et al., 2005].

7. Overall, we notice that the values of the fracture energy density obtained in our dynamic models are comparable with values reported in previous studies [see Guatteri and Spudich, 2000; McGarr et al., 2004 for a review]; we emphasize here that seismological estimates ($E_G \sim 10^2$ to $10^3$ J/m$^2$) can be an overestimate due to low-pass filtering of the seismograms [Spudich and Guatteri, 2004].

To finish, the dependences of the fracture energy density on rupture velocity, on cumulative fault slip, and on dynamic stress drop as discussed above depend on the adopted governing equation (the choice of which is still matter of a lively debate [e.g., Bizzarri and Cocco, 2006b] and, what is more, on the rupture regime (cracklike or pulselike; as well as sub- or supershear speed). Moreover, these dependencies appear to be in favor of the idea [see also Okubo and Dieterich, 1986; Abercrombie and Rice, 2005] that $E_G$ should not be regarded as an intrinsic material parameter.

Appendix A: Effects of the Nonzero-Offset Correlation Analysis

In the grid-by-grid analysis presented in sections 5–7 we have considered a zero-offset distance correlation analysis. In other words, we have considered the value of the fracture energy density and of the other physical quantities in the same fault node. Here, we consider a nonzero-offset distance correlation, in which the observables are defined in different points of the fault plane.

Let us consider two spatially distributed 2D arrays, $X \equiv \{x_{i,k}\}$ and $Y \equiv \{y_{i,k}\}$ (with $i = 1, \ldots, i_{end}$ and $k = 1, \ldots, k_{end}$), characterized by their average values, $\langle X \rangle$ and $\langle Y \rangle$, respectively, and by their standard deviations, $\sigma_X$ and $\sigma_Y$, respectively. We will consider the normalized covariance between them, $C$, defined as [e.g., Goovaerts, 1997]

\[ C = \frac{E(\langle X - \langle X \rangle \rangle (\tilde{Y} - \langle Y \rangle))}{\sigma_X \sigma_Y}, \quad (A1) \]

where $E(\cdot)$ is the expected value operator and $\tilde{Y}$ denotes the array $Y$ translated by a vector $h = ((\alpha - 1) \Delta x, (\beta - 1) \Delta x) = ((\alpha - 1), (\beta - 1)) \Delta x$. More specifically, we compute $C$ of (A1) as

\[ C_{\alpha, \beta} = \sum_{i=1}^{i_{end}-\alpha+1} \sum_{k=1}^{k_{end}-\beta+1} \frac{(x_{i,k} - \langle X \rangle) (y_{i+\alpha-1,k+\beta-1} - \langle Y \rangle)}{(i_{end} - \alpha + 1) (k_{end} - \beta + 1) \sigma_X \sigma_Y} \quad (A2) \]
Results of the nonzero offset distance analysis for the model reported in Figures 3, 4, and 5. The normalized cross-correlation, calculated as in equation (A2), is reported as a function of spatial offset between fracture energy density and (a) rupture velocity, (b) total cumulative fault slip, and (c) dynamic stress drop.

Figure A1. Results of the nonzero-offset distance analysis for the model reported in Figures 3, 4, and 5. The normalized cross-correlation, calculated as in equation (A2), is reported as a function of spatial offset between fracture energy density and (a) rupture velocity, (b) total cumulative fault slip, and (c) dynamic stress drop.

with \( \langle X \rangle = \frac{1}{l_{\text{end}} k_{\text{end}}} \sum_{i=1}^{l_{\text{end}}} \sum_{k=1}^{k_{\text{end}}} x_{i,k} \) and \( \sigma_X = \sqrt{\frac{1}{l_{\text{end}} k_{\text{end}}} \sum_{i=1}^{l_{\text{end}}} \sum_{k=1}^{k_{\text{end}}} (x_{i,k} - \langle X \rangle)^2} \) (and analogous expressions for \( Y \)). The translation vector \( h \) defines a spatial offset distance \( h = \sqrt{(\alpha - 1)^2 + (\beta - 1)^2} \Delta x \) and an azimuth angle \( \varphi = \arctan ((\beta - 1)/(\alpha - 1)) \).

The pair \((\alpha, \beta) = (1, 1)\) corresponds to a zero-offset distance, so that \( C_{\alpha,\beta} \) becomes the autocorrelation function \( \chi = E((X - \langle X \rangle)(Y - \langle Y \rangle))/\sigma_X \sigma_Y \). The \( C_{\alpha,\beta} \), which is also known as a correlogram \([\text{Goovaerts}, 1997]\), represents the linear dependency between the two variables \( X \) and \( Y \), and it varies between \(-1\) and \(1\) [see also Song et al., 2009]. The evaluation of \( C_{\alpha,\beta} \) for different values of \( h \) (i.e., for different values of \( \alpha \) and \( \beta \)) enables us to quantify the potential spatial coherence between spatially varying variables \( X \) and \( Y \).

For the present purposes, we associate \( X \) to \( E_G \) and \( Y \) alternatively to \( v_r \), \( u_{\text{tot}} \), and \( \Delta \tau_d \). Results pertaining to the model reported in Figures 3, 4, and 5 are plotted in Figure A1, where values of \( C_{\alpha,\beta} \) corresponding to the same value of spatial offset distance \( h \) are averaged [see Bizzarri et al., 2010, Figure 11b]. We can clearly see that maximum spatial correlation exists at the zero-offset distance for all of the physical observables, \( v_r \) (Figure A1a), \( u_{\text{tot}} \) (Figure A1b), and \( \Delta \tau_d \) (Figure A1c). For increasing spatial offset \( C_{\alpha,\beta} \) decreases reaching a minimum for a value of \( h \) nearly equal to 2.3 km. This behavior recalls that obtained by Song et al. [2009, Figure 4], that relates slip to peak slip velocity and rise time. The slopes of the three curves reported in Figure A1 are slightly different, with a roll-off in the case of slip (Figure A1b).

The maximum spatial correlation existing at zero-offset distance corroborates the same point, grid-by-grid analysis presented in sections 5 to 7.

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A. Bizzarri, Istituto Nazionale di Geofisica e Vulcanologia, Sezione di Bologna, Via Donato Creti 12, I-40128 Bologna, Italy. (bizzarri@bo.ingv.it)