Appendix C

Convergence and stability conditions for 2 – D fault models

In this Appendix we briefly summarize the convergence and stability conditions that have to be satisfied in order to correctly resolve the dynamic traction evolution and the slip velocity behavior within the cohesive zone during the propagation of a dynamic rupture obeying to rate– and state–dependent friction.

In numerical analysis the term "convergence" is essentially used to quantify how the solution of a problem is good. Such an evaluation is composed by two subsequent steps: first, one have to quantify the *consistence* (i. e. how the discretized equations well represent the (physical) problem to be solved); second, one have to check the *stability* (i. e. if the numerical, approximated solution is unique, non divergent and how it is close to the exact one, in particular as the grid is refined or remeshed). The first step leads to the continuum approximation problem, while the second one leads to a comparative analysis. It is well–know that analytical solution of the fully dynamic, spontaneous problem do not exist. Therefore the comparison may be done using different numerical methods (Bizzarri et al., 2001), different fault boundary conditions (Andrews, 1999) or comparing the solution obtained with different grid sampling (see the Section Convergence analysis in the present work). The latter approach represent also an useful tool to evaluate the behavior of the numerical noise and the changes in the truncation errors.

C.1. BIE with slip – weakening law

In order to obtain physically acceptable solutions by using the BIE method, several conditions have to be satisfied to determine a correct discretization of the dynamic problem. We remind that equation (2.4) is valid under the assumption that $Dx \geq v_P\ Dt$. This condition is typical of all the boundary integral equation methods. According to Andrews (1985) linear analysis considerations yields

$$\Delta x < -\frac{\mu\, v_P}{v_S \dfrac{\mathrm{d}}{\mathrm{d}t}\tau}$$

or for an in – plane crack, examined in this study:

$$\frac{l_c^{(\mathrm{II})}}{\Delta x} > \frac{2}{\pi}\frac{a-1}{\sqrt{a}}(1+S)^2$$

where a is the square root of the v_P / v_S ration ($a^2 \equiv v_P / v_S$), S is the strength parameter previously defined, and l_c is the critical half length (whose expression is derived in Appendix D).

Moreover, in order to resolve the cohesive zone, that is to verify that the discretization guarantees a sufficient number of points in the breakdown zone, it is required that $Dx < d_0$.

We emphasize that these conditions are quite important to verify the dynamic solutions. In particular, the last condition is very important to properly take into account the adopted constitutive law. Different simulations have shown that a very suitable choice is $d_0 \cong 6 \div 7\ Dx$. It is important also to point out the previous conditions are simultaneously satisfied only when $t_u - t_f \geq m\, v_P / v_S$.

C.2. FD with slip – weakening law

For the slip – weakening constitutive law there is not a definition of critical stiffness k_{cr} as in the rate – and state – dependent friction laws framework (see next Section). The first conditions that has to be satisfied is the correct resolution of the cohesive zone

$$Dt << \frac{d_0}{< v >_{T_b^{eq}}} \quad \text{or, alternatively,} \quad Dx << \frac{1}{w_{\mathrm{CFL}}} \frac{v_S}{< v >_{T_b^{eq}}} d_0 \qquad \text{(C.2.1)}$$

The second requirement is that the spatial and time steps are coupled by the general condition (Andrews, 1985; Fukuyama and Madariaga, 1998; Bizzarri et al., 2001, among many different others)

$$Dx \geq v_P\, Dt, \qquad \text{(C.2.2)}$$

which states that no coupling exists between first neighbors.

C.3. FD with rate – and state – dependent friction laws

The first condition that has to be satisfied is a requirement introduced by Rice (1993) to demonstrate that artificial numerical complexity can appear if the medium is not correctly discretized as a continuum; it depends on the fault geometry and on the boundary conditions. In full of generality it can be expressed as $k_{diag} >> k_{cr}$, where k_{diag} is the diagonal term of stiffness matrix and k_{cr} is the critical stiffness. The requirement $k_{diag} >> k_{cr}$ corresponds to impose that locally each single element of the discretized fault is conditionally stable (Scholz, 1990). This avoids that a single point may fail independently of the neighbors (artificial complexity and numerical noise) and guarantees that the discrete medium can be considered as a continuum. The local stiffness is expressed as $k_{diag} = 1/C$, where C is the local compliance (Andrews, 1985; Bizzarri et al., 2001). C represents the proportionality constant between instantaneous traction and dynamic slip and in our 2 – D FD fault model is $C = 3½\ v_S r\ /\ (8w_{CFL} Dt)$, where w_{CFL} is the Courant – Friedrichs – Levy (CFL) ratio, that relates Dx to Dt ($w_{CFL} = v_S\ Dt / Dx$; see Fukuyiama and Madariaga, 1998; Bizzarri et al., 2001). The critical stiffness can be expressed as $(b - a)\ s_n^{eff} / L$ (Ranjith and Rice, 1999), where the constitutive parameters a, b, s_n^{eff} and L have been assigned. When $k_{diag} = k_{cr}$, we have the critical grid size:

$$D t^* = \frac{v_S\, r\, L}{(b-a)\, s_n^{eff}} \frac{8}{\sqrt{3}} \mathbf{w}_{\mathbf{CFL}} \quad \text{or, alternatively,} \quad D x^* = \frac{v_S^{\,2}\, r\, L}{(b-a)\, s_n^{eff}} \frac{8}{\sqrt{3}} \qquad \text{(C.3.1)}$$

The Rice's condition can be therefore expressed as:

$$Dt << Dt^* \quad \text{or, alternatively, as} \quad Dx << Dx^* \qquad \text{(C.3.2)}$$

For our purposes we have to verify also that the numerical integration is able to correctly resolve time scales typical of the dynamic evolution of the state variable. Following Ohnaka and Yamashita (1989) and Cocco and Bizzarri (2002), we define here the equivalent breakdown zone duration (or time) as T_b^{eq} and the equivalent cohesive zone size as X_b^{eq}. We remind here that during the breakdown time duration and over the cohesive zone distance the friction decreases from the maximum yield value to the kinetic level and, according to our interpretation, such a dynamic behavior is controlled by the state variable evolution. Therefore, the requirement of resolution of this characteristic time duration and spatial scale consists to impose the following
conditions:

$$D t << \frac{d_0^{\,eq}}{< v >_{T_b^{eq}}} \quad \text{or, alternatively,} \quad D x << \frac{1}{w_{\mathrm{CFL}}} \frac{v_S}{< v >_{T_b^{eq}}} d_0^{\,eq} \qquad \text{(C.3.3a)}$$

or, in a different way:

$$D t << w_{\mathrm{CFL}} \frac{1}{v_S} \frac{v_{crack}}{< v >_{T_b^{eq}}} d_0^{\,eq} \quad \text{or, alternatively,} \quad D x << \frac{v_{crack}}{< v >_{T_b^{eq}}} d_0^{\,eq} \qquad \text{(C.3.3b)}$$

where v_{crack} is the crack speed propagation and $<v>_{T_b^{eq}}$ is the average slip velocity calculated within the equivalent breakdown zone time (see Bizzarri et al., 2001 for further details).

Finally, the spatial and time steps are coupled by the general condition (Andrews, 1985; Fukuyama and Madariaga, 1998; Bizzarri et al., 2001, among many different others)

$$Dx \geq v_P \, Dt, \qquad \text{(C.3.4)}$$

which states that no coupling exists between first neighbors. This condition is common to Boundary Integral Equation (BIE) methods and to Finite Difference (FD) approaches.

In practice, in our simulation we adopt the following procedure. First, we choose the spatial discretization in order to satisfy the Rice's condition concerning the validity of the continuum approximation: $Dx << 8\ v_S^{2} r L\ /3½\ (b - a) s_n^{eff}$. Thus, we choose the spatial time step in order to resolve the cohesive zone

$$Dt << d_0^{\,eq} \ / <v>_{T_b^{eq}} \cong L \ln(v_0/v_{init}) \ / <v>_{T_b^{eq}}$$

and to satisfy the first neighbors decoupling condition ($Dt \leq v_P \, Dx$). Therefore, we assign Dx and Dt (consequently, we fix w_{CFL}) in a way that all the three convergence and stability disequations are satisfied. However, we emphasize that fulfilling these set of relations does not specify how much Dx has to be smaller than $8 v_S^{2} r L\ /3½\ (b - a) s_n^{eff}$ or, analogously, how much Dt has to be smaller than $d_0^{\,eq} \ / <v>_{T_b^{eq}}$. It has been shown in Bizzarri and Cocco (2003), their Figures 4 and 5, that even if all three conditions are correctly satisfied the behavior of the solutions is quite different if we further refine our discretization or if we change the CFL ratio. The only way to choose the best discretization is to check the time histories, phase diagrams and slip–weakening curves for different discretization cases, like in Figures 4 and 5 of Bizzarri and Cocco (2003). Then we chose the first smallest values of *Dx* and *Dt* below which all curves become independent on the adopted numerical discretization. A further analysis may be necessary to tune the value of the CFL ratio.