Appendix D

Correspondence between fracture energy and others parameters in different constitutive models

In section 2.5 we have derived the relations between the constitutive parameters of Dieterich – Ruina (DR) and those of the slip – weakening (SW) governing equations. Here we propose the correspondence existing between the fracture energy G and the critical half–length l_c in the framework of different constitutive laws.

In general the fracture energy G can be expressed as (Ida, 1972; Aki, 1979):

$$G = \frac{1}{2} \int_{0}^{+\infty} \left(t(u) - t_f \right) \mathrm{d}u \tag{D.1}$$

In the SW friction model this relation gives:

$$G^{(\mathrm{SW})} = (t_u - t_f) \frac{d_o}{4} \tag{D.2}$$

whereas for the rate– and state–dependent friction laws (using the approximate expression for the friction (2.14)) it results:

$$G^{(DR)} = \frac{1}{2}\left(a\, s_n^{\,eff} \ln\left(\frac{< v >}{v_0} \right) + b\, s_n^{\,eff} \ln\left(\frac{v_0}{v_{init}} \right) \right) d_0^{\,eq} - \frac{1}{4}\frac{b\, s_n^{\,eff}}{L} d_0^{\,eq^2} \qquad \text{(D.3)}$$

that, taking into account equation (2.13), can be rewritten as

$$G^{(DR)} = \frac{1}{2}\left(a\, s_n^{\,eff} \ln\left(\frac{< v >}{v_0} \right) + b\, s_n^{\,eff} \ln\left(\frac{v_0}{v_{init}} \right) - \frac{b\, s_n^{\,eff}}{2}\left(\ln\left(\frac{v_0}{v_{init}} \right) \right)^2 \right) d_0^{\,eq} \qquad \text{(D.4)}$$

Let us emphasize that both with the SW and with the DR law the energy released in the rupture process increases linearly with the characteristic length and with the difference between the peak value reached by the friction and the final kinetic level.

In both models it exists a minimum extension, also called critical patch size, below which a crack cannot propagate. Andrews (1976a, 1976b), starting from considerations based on the energy balance principles for planar shear fractures, has derived that both for mode III (anti – plane) and mode II (in – plane) cracks the propagation is possible only when the rupture has an half – length greater then a threshold value, whose expression results:

$$l_c^{(\mathrm{III})} = \frac{m}{p} \frac{t_u - t_f}{(t_0 - t_f)^2}\, d_0 \qquad \text{(D.5a)}$$

for anti–plane cracks (Andrews, 1976a) and

$$l_c^{(\mathrm{II})} = 2\frac{m}{p} \frac{l + m}{l + 2m} \frac{t_u - t_f}{(t_0 - t_f)^2}\, d_0 \qquad \text{(D.5b)}$$

for in – plane ruptures (Andrews, 1976b).

An analogous expression has been introduced for the rate – and state – dependent friction laws. Studies performed about the stability of spring slider systems with only one degree of freedom (Dieterich, 1979a, 1981; Rice and Ruina, 1983; Gu et al., 1984, between among others) have shown that the slip can accelerate to the instability if the stiffness k of the mass spring system is less than a critical value, expressed by the relation

$$k_{cr} = k \frac{s_n^{eff}}{L} \tag{D.6}$$

where k is a constant depending on the parameters of the adopted constitutive law and on the conditions under which the experiment is performed. In particular, in the quasi–static approximations (both for the Ruina – Dieterich law (Gu et al., 1984) and for the Dieterich – Ruina governing equation (Ranjith and Rice, 1999) it results $k = b - a$. Combininig the results for a spring–slider system with the solutions for the displacement due to a crack in an elastic medium, assuming that the properties and the conditions in the centre of the crack are adequate to represent of the entire rupture, we obtain (Dieterich, 1986, 1992):

$$k = m \frac{h}{l} \tag{D.7}$$

where h is a factor depending by the rupture geometry (Dieterich, 1992) and l is the half – length — or the radius — of the crack.
Comparing (D.6) and (D.7) it is possible to write an approximate relation for the minimum half – length which gives an instable behavior:

$$l_c = h\, m \frac{L}{k\, s_n^{eff}} \tag{D.8}$$