1. Source dynamics and constitutive models

1.1. The coseismic processes

An earthquake can be regarded as a propagating fracture originating from the accumulation of the tectonic load. In this representation it a consequence of the general theory of elastic rebound by Reid (1910). Such a fracture is analytically described as a deviation from the linear perfectly elastic behaviour, it nucleates and propagates on a fault surface, the geometric locus where the Hooke' s law is no longer valid. The surface is an idealization, because physical processes can take places within a zone of non zero width, but it physically reasonable because the lateral dimensions are paramount.

A further approximation made in source dynamics is that of planar fault. The fault surface is, in general, not planar and recent papers (for instance

Aochi et al. 2000) developed numerical methods to account such effects. In the source mechanics we can categorize the ruptures considering the dependencies on independent variables: we identifies 1 – D models as problem in which the solutions depend only on the temporal coordinate. These models, known as spring – sliders are obviously crude simplification of the behaviour of the fault, and are useful to consider only general phenomena, as nucleation, stability of the system and triggering effects. We will discuss such a model in Chapter 5. A more complete description of a planar fault is represented by 2 – D models, in which the solutions depend on one spatial coordinate and on the time. Within these model we can distinguish between tensile fracture (a.k.a. mode I) and shear fractures (a.k.a. mode II, or in – plane, and mode III, a.k.a. anti – plane). In Chapters 2 and 3 we will present such models. A successive step of fault approximation is the pseudo 3 – D models, a.k.a. mixed mode: we have now two non null components of the displacement, but each component obeys to pure mode II and pure mode III, respectively. A good example of such an approximation is in Andrews (1985). The final step, the more realistic description of a fault plane are 3 – D models, as described in Chapter 4.

In this Chapter we briefly introduce the reader to the dynamic problem, by discussing the importance of fracture criteria and of the constitutive models used in the following of the Ph.D. Thesis.

1.2. The dynamic problem

In this Ph.D. Thesis we solve the fundamental elastodynamic equation, i. e. the Second Law of dynamics for continuum media. While classical mechanics is focused on the motion of a medium considered as a whole, the mechanic of continua considers the shape modifications that media suffer during some physical processes. Instead of considering the instantaneous position of a generic point $(x(t),y(t),z(t))$, we use the displacement field $\mathbf{u}(\mathbf{x}(t),t)$, analytically defined as the difference existing between the position $\mathbf{x}'$ of the point moved and its position $\mathbf{x}$ before the displacement; $\mathbf{u} = \mathbf{x}' - \mathbf{x}$.

The fundamental elastodynamic equation assumes the form:

$$r\ \ddot{u}_i = f_i + s_{ij,j} \tag{1.1}$$

where r is the volumetric density of mass of the medium to which the considered point is, **u** the displacement, f_i the i – component of the applied external body forces and s_{ij} the component of the stress tensor. This is the Cauchy stress, that is that two particles in a continuum medium interact only trough the surface separating them. Equation (1.1), in which the Einstein' s index summation is assumed, is an analytical representation of displacement field due to variation of stress variation and application of body forces.

The solution of (1.1) may be done in different ways. One can assign the slip distribution on the fault surface: in such a way we have *kinematic models*. The main topic is the study of the displacement discontinuity and the fault evolution is represented by unilateral or bilateral motion (a good end – member of this class is the rectangular Haskell' s model). In kinematic models we consider only the time evolution of the rupture front, neglecting the dynamics of faulting; they are good for the study of the long period seismic waves, with wavelength l greater than the total fault length l_{fault}. However, there are important limitations: constant dislocation is inadmissible, the strain energy at the crack tip is unbounded and the stress drop is infinite.

If we impose a finite energy flaw at the tip we have *dynamic models*: the slip is not prescribed a priori, but is calculated from the stress distribution and from the fault strength S. The shear stress drops inside the crack (after the nucleation processes), increases the stress outside the cracked area (near the tip) and tends to facilitate further grow of the rupture. In this way the motion is determined by the fracture criterion (and eventually by the assumed governing law on the fault surface). Dynamic problems are characterized by assuming mixed boundary conditions on the fault surface: the slip outside the crack tip and the stress tensor components inside the crack tip. Kinematic models are mainly interested on seismic waves or travel times, and take allows to infer material properties, like elastic constant (l and m), density (r), waves velocities (v_P, v_S). Dynamic models consider wave dynamics and allow for the inference of source parameters, like stress levels (for instance t_0, t_u and t_f, see next Section), characteristic length (L or d_0, see Sections 1.4 and 1.5), governing equations.

In this work we consider the forward problem, i. e. the determination of the displacement and stress distribution starting from assigned source parameters. This is an important tools for understanding real – world data and for elucidate physical properties of constitutive laws, as detailed physical mechanisms described and salient features of a fault model. In this contest we imagine that the source parameters have been fixed and have been derived from a previous kinematic study of the fault. In fact kinematic models can be regarded form our point of view as a tool for processing data and to extract source parameter form observables.

The forward modelling can be schematized as the product of five conceptual steps. The first step is the configuration, i. e. the assignation of the fault geometry, in terms of orientation and of its characteristic (planar or not), and of the fault system, as multiple segments or multiple faults. The second one is the medium, i. e. the properties of the medium surrounding the fault(s): density, velocity structure, anisotropy, attenuation and all phenomena related to the waves propagation in continua. The third step is the fault state, expressed as initial conditions on the fault (like initial slip, initial slip velocity, initial state variable, pre – stress, …) and initial conditions outside the fault (like tectonic load, state of the neighbouring faults). This is because a fault is not an isolated system. The fourth step is represented as fault frictional properties, in terms of rheology on the fault and of constitutive laws (see next Section). The last step is numerical simulations: choice of the numerical method, implementation of the fault boundary condition and convergence analysis (see Appendixes C and F). In this Ph.D. Thesis we will focus on the latter three steps mainly.

Because we consider the fully dynamic, spontaneous problem (i. e. the crack tip speed, or analogously the crack tip enlargement velocity, is not assigned a priori but is a solution of the problem), no analytical solution can be derived in closed form. We have to treat equation (1.1) numerically. The intrinsically non – linear process of dynamic crack propagation has been studied in last decades by using a wide range of methods (lattice differential equations, partial different equations, cellular automata; see Senatorski, 1994 for a complete review) and numerical models (Boundary Integral Equation (BIE) methods, Finite Difference (FD) approaches, Finite Elements (FE) and Boundary Element (BE) schemes). While automata facilitate rapid numerical computation, continuum models allow a more direct associations between model parameters, like characteristic length and time scales, elastic constants and observations or real – world phenomena. We will describe our numerical procedures, BIE and FD, for 2 – D in – plane fault in Chapter 2, FD method for 3 – D faults in Chapter 4 and finally the simple spring – block model in Chapter 5.

1.3. Fracture criteria and governing models

A rupture can be described in two ways. The first, known as *crack model*, consider the energy dissipation at the crack edge (or crack tip) as paramount. The second one is *friction model*, in which the effects at the edges are not explicitly considered, but explicitly allows for the calculation of the evolution of the stress tensor components in terms of material properties of the fault.

The solution of the fully dynamic problem arise form two ingredients: fracture criteria and friction laws. Fracture criteria are mathematical conditions that specify at a given fault point and at a given time if there is a rupture or not. They can be expressed in terms of energy, in terms of maximum frictional resistance and so no. They are essentially based on the Beniof' s (1951) hypothesis that a fracture occurs when the stress in a volume reaches the rock strength, or on the Reid' s (1910) statement that a fracture takes places when the stress attain a value greater than rock can endure. Fracture criteria are not able to inform us about conditions before a rupture, nor during nor after; they simply express the failure limit. We refer to Bizzarri (1998) to an exhaustive review of fracture criteria.

On the contrary, constitutive (or governing) models are analytical relations between the components of the stress tensor and physical observables, like the slip, the slip velocity, the state variable, etc. The fracture is not instantaneous: it is not occurring at the same time for all the points if the medium, even is the medium is macroscopically homogeneous, it is microinhomogeneous (Kostrov and Das, 1988): the critical state is reached only within a patch, that propagates stress and cause a redistribution of stresses in the surrounding medium. Such a redistribution travel a v_P at maximum. Constitutive laws allow us to model the whole failure process, starting from the initiation up to the arrest. In full of generality a governing models can be expressed writing the friction (i. e. the modulus of shear stress components on the fault surface) as:

$$t = m(u, (\mathrm{d}/\mathrm{d}t)u, \{\mathrm{Y}\}_{i = 1,\dots, n}, T, H, l_c, h, g, C_e)\ s_n^{eff}(s_n, p_{fluid}) \qquad (1.2)$$

where u is the slip modulus, $(\mathrm{d}/\mathrm{d}t)u$ is the slip velocity modulus, $\{\mathrm{Y}\}_{i = 1,\dots, n}$ are the n state variables associated to corresponding n evolution equations (see Section 1.5). T is the temperature, H is the humidity (Dieterich and Conrad, 1984), l_c is the characteristic length of surface in contacts (accounting for roughness and topography of asperity contacts), h is the hardness, g is the fault gouge (accounting for surface consumption and subsequent gouge formation), C_e is the chemical environment, and $s_n^{eff}(s_n, p_{fluid})$ is the effective normal stress, dependent on the regional, reference normal stress and fluid pressure (see for instance Andrews, 2003). The direct dependence of fault friction on normal stress expresses the second law of Amonton.

A constitutive law is, de facto, a boundary condition that controls earthquake dynamics and its complexity. It is the macrophysic of rheology, in the sense that it describe continuum mechanics, introducing macrosocopic relations between stress, strain and their time derivatives. In general constitutive laws derive from geological and geophysical observations and data, as well as from laboratory experiments, as friction experiments (stick – slip) and fracture of intact rocks. They do not consider explicitly the microphysic of rheology, i. e. the processes at atomic scale and lattice level, but can be regarded as an average of behaviour of mycrostructural observations.

1.4. Slip – dependent constitutive laws

The simplest constitutive law is represented by assuming that only two frictional level exist: the maximum (upper or yield) shear stress t_u and the final (kinetic) shear stress t_f. The first one is the higher value of friction above which the material is not able to accommodate with deformations and fail; the second one is the residual level at which the material slides. From such a simplified governing model (in which the fracture criterion is the achievement of the upper value) we have the stress drop, formally the static stress drop: $Dt_s = t_u - t_f$. The limitation of this model is that the stress release is instantaneous and there is a singularity in the stress tensor component at the tip. There is no any characteristic time for the release process, nor a characteristic length. However it is particularly useful for the introduction of the strength concept: we define the strength parameter as the difference existing between the stress excess and the stress drop (Das and Aki, 1977a, 1977b):

$$S \underset{\mathrm{df}}{=} \frac{t_u - t_0}{t_0 - t_f} \tag{1.3}$$

where t_0 is the initial shear stress. It is important to note that the adimensional *S* parameter, that represent the ability for a fault to fail: an low value of S indicates a fault that is ready to fracture, it is not the dimensional fault strength S_{fault}, that is in general expressed as

$$S_{fault} = m\, s_n^{eff} \tag{1.4}$$

where the analytical form of m describe the adopted constitutive law (see equation (1.2)).

To avoid singularities and to make model able to describe more correctly the stress release and the physical processes that take place during the breakdown process, Barenblatt (1959) introduce the concept of cohesive zone, that guarantees that the energy flux is bounded. The governing law that directly originates from the cohesive zone concept is the slip – weakening law.

In its classical formulation, due essentially to Ida (1972) and afterwards adopted in numerical simulation of a 2 – D pure in – plane fault by Andrews (1976a, 1976b), the slip – weakening (SW) expresses a linear decay of friction with increasing displacement:

$$t = \begin{cases} t_{\mathbf{u}} - (t_{\mathbf{u}} - t_{\mathbf{f}}) \dfrac{\mathbf{u}}{\mathbf{d}_0} & , \mathbf{u} < \mathbf{d}_0 \\ t_{\mathbf{f}} & , \mathbf{u} \geq \mathbf{d}_0 \end{cases} \tag{1.5}$$

We will show results of a 2 – D pure in – plane rupture obeying to SW in Chapter 2. As discussed in the following Chapters, the SW law has a characteristic length d_0 associated with the breakdown processes: the stress degrades from the yield stress to the kinetic level during a breakdown zone time T_b and over a breakdown zone distance X_b. It does not include any hardening effects: the material is intact if total traction is lower that the upper stress; if is at that level the rupture starts: that is the fracture criterion in this case, formally a threshold condition for stress. When the rupture is already started the friction is expressed as a function of the developed displacement as in equation (1.5). The crack never stop in the classical SW model; it obviously is an approximation, because in this case the total seismic moment would be unbounded, as the cracked area would be infinite. Frictional heterogeneities discussed in Chapter 3 allow for a more realistic modelling of fault, including rupture arrest and healing.

As discussed before, the SW law has been introduced theoretically. Laboratory experiments made by Ohnaka and coworkers (see for instance Ohnaka and Yamashita, 1989) measured the total friction acting along a preexisting fault and they interpreted results as SW. More exactly it is a slip dependent law, in which there is an initial slip – hardening phase (i. e. increasing friction with increasing slip) and then a slip – weakening phase, during which the traction degrade exponentially with increasing slip to an asymptotic frictional level. Because of its fascinating simplicity, the SW model has been recently used to model real – world events (Ide and Takeo, 1997; Guatteri and Spudich, 2000, Peyrat et al., 2002). We will emphasize the characteristics of Sw law in Chapter 2.

1.5. Rate – and state – dependent friction laws

An intensive laboratory works made by Dieterich and coworkers show that stick – slip experiments data can be interpreted in the framework of the so – called governing laws with memory, or rate – and state – dependent friction laws. The original analytical formulation is essentially due to Dieterich (1976) and Ruina (1980, 1983). In this law the friction is assumed to be dependent on the slip velocity and one only one state variable, that account for the previous failure episodes and in general represent the previous history of the fault and of the asperity contacts (Bowden and Tabor, 1950, 1964; Rabinowicz, 1965).

$$\begin{cases} t\,(t) = s_n\, F\big(\, v(t), Y\,(t)\,\big) & (1.6a) \\ \dfrac{\mathrm{d}}{\mathrm{d}t} Y\,(t) = s_n\, G\big(\, v(t), Y\,(t), L\,\big) & (1.6b) \end{cases}$$

The first equation is formally the constitutive law, while the second one is the evolution equation for the state variable. The Dieterich model (to which we refere in the following as the Dieterich in reduced form) is:

$$t = t_* - A \ln\left(\frac{v_*}{v} + 1 \right) + B \ln\left(\frac{F\, v_*}{L} + 1 \right) \qquad (1.7a)$$

$$\frac{\mathrm{d}}{\mathrm{d}t} F = 1 - \frac{F\, v}{L} \qquad (1.7b)$$

where A accounts for the direct effect, B for the evolving effect and L in the characteristic length for the state variable evolutions. F physically represent the average contact time between fault surface asperities. Equation (1.7b) includes true ageing (Perrin et al., 1995).

A further analytical representation of friction is the Ruina model:

$$t = t_* + q + \mathbf{A} \ln\left(\frac{\mathbf{v}}{\mathbf{v}_*} \right) \qquad (1.8a)$$

$$\frac{\mathrm{d}}{\mathrm{d}t} q = -\frac{v}{L}\left[\, q + B \ln\left(\frac{v}{v_*} \right) \right] \qquad (1.8b)$$

We can redefine the Ruina' s state variable in order to rewrite equation (1.8a) in the same constitutive form as in the Dieterich friction (equation (1.7a)). In such a way a simple algebra leads to the so – called Ruina – Dieterich model:

$$t = \left[m_* + \mathbf{a}\, ln\left(\frac{\mathbf{V}}{\mathbf{V}_*}\right) + \mathbf{b}\, ln\left(\frac{Q\mathbf{V}_*}{\mathbf{L}}\right)\right] s_{\mathbf{n}}^{\mathbf{eff}} \qquad (1.9a)$$

$$\frac{\mathbf{d}Q}{\mathbf{dt}} = -\frac{Q\mathbf{V}}{\mathbf{L}}\, ln\left(\frac{Q\mathbf{V}}{\mathbf{L}}\right) \qquad (1.9b)$$

These laws allow for the modelling of the fault nucleation (Dieterich, 1992, 1994), hardening stage and they have been recently used to model real earthquakes (Guatteri et al., 2002). The rate – and state – dependent friction laws are the second class of constitutive models that we use in this Ph.D. Thesis: we will discuss results in Chapters 2, 3 and 5.

1.6. Are the models complete? Additional functional dependencies

The above – mentioned constitutive models are only the historical ones, because in the literature different governing models have been proposed, derived, adopted in numerical simulation. Let us mention the rate – dependent friction laws (Burridge and Knopoff, 1967; Carlson and Langer, 1989; Madariaga and Cochard, 1994) in which the friction depend only on the slip velocity; the slip – and rate – dependent (Cochard and Madariaga, 1995), in which the friction depends on the displacement and on the slip velocity. Perrin et al. (1995) used different regularization of (1.7) to model the short slip duration and the so – called self – healing phenomenon; we will discuss these aspects in Chapter 3.

All these models consider only one state variable, but a study by Weeks and Tullis (1985) consider a formulation with two different variable. Recent studies have been made to incorporate in the constitutive equations other physical observables, enumerated in equation (1.2): Andrews (2003) propose a model in which frictional heating causes fluid migration and therefore a n accommodation of normal stress. Linker and Dieterich (1992) introduced a model to incorporate the variations of normal stress in the evolving equations.

In spite of the possible theoretical complications of the models, to be more and more realistic, there is the fundamental question to be solved, that is how to scale the laboratory data to the real – world phenomena. In particular, the simple law described before are experimentally derived, but only in a limited range of slip velocities (up to centimetres, see Mair and Marone, 1996). What is more we don' t know how large is the characteristic length for real faults. We will discuss these problems in the following Chapters and we will present possible future works in this research field in Sections 6.2.